

ON CR PANEITZ OPERATORS AND CR PLURIHARMONIC FUNCTIONS

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ABSTRACT. Let $(X, T^{1,0}X)$ be a compact orientable embeddable three dimensional strongly pseudoconvex CR manifold and let P be the associated CR Paneitz operator. In this paper, we show that (I) P is self-adjoint and P has L^2 closed range. Let N and Π be the associated partial inverse and the orthogonal projection onto $\text{Ker } P$ respectively, then N and Π enjoy some regularity properties. (II) Let $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_0$ be the space of L^2 CR pluriharmonic functions and the space of real part of L^2 global CR functions respectively. Let S be the associated Szegő projection and let τ, τ_0 be the orthogonal projections onto $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_0$ respectively. Then, $\Pi = S + \bar{S} + F_0$, $\tau = S + \bar{S} + F_1$, $\tau_0 = S + \bar{S} + F_2$, where F_0, F_1, F_2 are smoothing operators on X . In particular, Π, τ and τ_0 are Fourier integral operators with complex phases and $\hat{\mathcal{P}}^\perp \cap \text{Ker } P$, $\hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$, $\hat{\mathcal{P}}_0^\perp \cap \text{Ker } P$ are all finite dimensional subspaces of $C^\infty(X)$ (it is well-known that $\hat{\mathcal{P}}_0 \subset \hat{\mathcal{P}} \subset \text{Ker } P$). (III) $\text{Spec } P$ is a discrete subset of \mathbb{R} and for every $\lambda \in \text{Spec } P$, $\lambda \neq 0$, λ is an eigenvalue of P and the associated eigenspace $H_\lambda(P)$ is a finite dimensional subspace of $C^\infty(X)$.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $(X, T^{1,0}X)$ be a compact orientable embeddable strongly pseudoconvex CR manifold of dimension three. Let P be the associated Paneitz operator and let $\hat{\mathcal{P}}$ be the space of L^2 CR pluriharmonic functions. The operator P and the space $\hat{\mathcal{P}}$ play important roles in CR embedding problems and CR conformal geometry (see [2] [3], [4]). The operator

$$P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$$

is a real, symmetric, fourth order non-hypoelliptic partial differential operator and $\hat{\mathcal{P}}$ is an infinite dimensional subspace of $L^2(X)$. In CR embedding

problems and CR conformal geometry, it is crucial to be able to answer the following fundamental analytic problems about P and \hat{P} (see [2] [3], [4]):

(I) Is P self-adjoint? Does P has L^2 closed range? What is $\text{Spec } P$?
 (II) If we have $Pu = f$, where f is in some Sobolev space $H^s(X)$, $s \in \mathbb{Z}$, and $u \perp \text{Ker } P$. Can we have $u \in H^{s'}(X)$, for some $s' \in \mathbb{Z}$?

(III) It is well-known (see Lee [9]) that $\hat{P} \subset \text{Ker } P$ and if X has torsion zero then $\hat{P} = \text{Ker } P$. It remains an important problem to determine the precise geometrical condition under which the kernel of P is exactly the CR pluriharmonic functions or even a direct sum of a finite dimensional subspace with CR pluriharmonic functions.

(IV) Let Π be the orthogonal projection onto $\text{Ker } P$ and let τ be the orthogonal projection onto \hat{P} . Let $\Pi(x, y)$ and $\tau(x, y)$ denote the distribution kernels of Π and τ respectively. The P' operator introduced in Case and Yang [2] plays a critical role in CR conformal geometry. To understand the operator P' , it is crucial to be able to know the exactly forms of $\Pi(x, y)$ and $\tau(x, y)$.

The purse of this work is to completely answer this questions. On the other hand, in several complex variables, the study of the associated Szegő projection S and τ are classical subjects. The operator S is well-understood; S is a Fourier integral operator with complex phase (see Boutet de Monvel-Sjöstrand [1], [7], [8]). But for τ , there are fewer results. In this paper, by using the Paneitz operator P , we could prove that τ is also a complex Fourier integral operator and $\tau = S + \bar{S} + F_1$, F_1 is a smoothing operator. It is quite interesting to see if the result hold in dimension ≥ 5 . We hope that the Paneitz operator P will be interesting for complex analysts and will be useful in several complex variables.

We now formulate the main results. We refer to section 2 for some standard notations and terminology used here.

Let $(X, T^{1,0}X)$ be a compact orientable 3-dimensional strongly pseudoconvex CR manifold, where $T^{1,0}X$ is a CR structure of X . We assume throughout that it is CR embeddable in some \mathbb{C}^N , for some $N \in \mathbb{N}$. Fix a contact form $\theta \in C^\infty(X, T^*X)$ compactable with the CR structure $T^{1,0}X$. Then, $(X, T^{1,0}X, \theta)$ is a 3-dimensional pseudohermitian manifold. Let $T \in C^\infty(X, TX)$ be the real non-vanishing global vector field given by

$$\begin{aligned} \langle d\theta, T \wedge u \rangle &= 0, \quad \forall u \in T^{1,0}X \oplus T^{0,1}X, \\ \langle \theta, T \rangle &= -1. \end{aligned}$$

Let $\langle \cdot | \cdot \rangle$ be the Hermitian inner product on $\mathbb{C}TX$ given by

$$\begin{aligned} \langle Z_1 | Z_2 \rangle &= -\frac{1}{2i} \langle d\theta, Z_1 \wedge \bar{Z}_2 \rangle, \quad Z_1, Z_2 \in T^{1,0}X, \\ T^{1,0}X \perp T^{0,1}X &:= \overline{T^{1,0}X}, \quad T \perp (T^{1,0}X \oplus T^{0,1}X), \quad \langle T | T \rangle = 1. \end{aligned}$$

The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces a Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$. Let $T^{*0,1}X$ be the bundle of $(0, 1)$ forms of X . Take $\theta \wedge d\theta$ be the volume form on X , we then get natural inner products on $C^\infty(X)$ and $\Omega^{0,1}(X) := C^\infty(X, T^{*0,1}X)$ induced by $\theta \wedge d\theta$ and $\langle \cdot | \cdot \rangle$. We shall use $(\cdot | \cdot)$ to denote these inner products and use $\|\cdot\|$ to denote the corresponding norms. Let $L^2(X)$ and $L^2_{(0,1)}(X)$ denote the completions of $C^\infty(X)$ and

$\Omega^{0,1}(X)$ with respect to $(\cdot | \cdot)$ respectively. Let

$$\square_b := \bar{\partial}_b^{*,f} \bar{\partial}_b : C^\infty(X) \rightarrow C^\infty(X)$$

be the Kohn Laplacian (see [7]), where $\bar{\partial}_b : C^\infty(X) \rightarrow \Omega^{0,1}(X)$ is the tangential Cauchy-Riemann operator and $\bar{\partial}_b^{*,f} : \Omega^{0,1}(X) \rightarrow C^\infty(X)$ is the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)$. That is, $(\bar{\partial}_b f | g) = (f | \bar{\partial}_b^{*,f} g)$, for every $f \in C^\infty(X)$, $g \in \Omega^{0,1}(X)$.

Let \mathcal{P} be the set of all CR pluriharmonic functions on X . That is,

$$(1.1) \quad \begin{aligned} \mathcal{P} = & \{u \in C^\infty(X, \mathbb{R}); \forall x_0 \in X, \text{ there is a } f \in C^\infty(X) \\ & \text{with } \bar{\partial}_b f = 0 \text{ near } x_0 \text{ and } \operatorname{Re} f = u \text{ near } x_0\}. \end{aligned}$$

The Paneitz operator

$$P : C^\infty(X) \rightarrow C^\infty(X)$$

can be characterized as follows (see section 4 in [2] and Lee [9]): P is a fourth order partial differential operator, real, symmetric, $\mathcal{P} \subset \operatorname{Ker} P$ and

$$(1.2) \quad \begin{aligned} P f &= \square_b \bar{\square}_b f + L_1 \circ L_2 f + L_3 f, \quad \forall f \in C^\infty(X), \\ L_1, L_2, L_3 &\in C^\infty(X, T^{1,0} X \oplus T^{0,1} X). \end{aligned}$$

We extend P to L^2 space by

$$(1.3) \quad \begin{aligned} P : \operatorname{Dom} P &\subset L^2(X) \rightarrow L^2(X), \\ \operatorname{Dom} P &= \{u \in L^2(X); P u \in L^2(X)\}. \end{aligned}$$

Let $\hat{\mathcal{P}} \subset L^2(X)$ be the completion of \mathcal{P} with respect to $(\cdot | \cdot)$. Then,

$$\hat{\mathcal{P}} \subset \operatorname{Ker} P.$$

Put

$$\mathcal{P}_0 = \{\operatorname{Re} f \in C^\infty(X, \mathbb{R}); f \in C^\infty(X) \text{ is a global CR function on } X\}$$

and let $\hat{\mathcal{P}}_0 \subset L^2(X)$ be the completion of \mathcal{P}_0 with respect to $(\cdot | \cdot)$. It is clearly that $\hat{\mathcal{P}}_0 \subset \hat{\mathcal{P}} \subset \operatorname{Ker} P$. Let

$$(1.4) \quad \begin{aligned} \tau : L^2(X) &\rightarrow \hat{\mathcal{P}}, \\ \tau_0 : L^2(X) &\rightarrow \hat{\mathcal{P}}_0, \end{aligned}$$

be the orthogonal projections.

We recall

Definition 1.1. Suppose Q is a closed densely defined self-adjoint operator

$$Q : \operatorname{Dom} Q \subset H \rightarrow \operatorname{Ran} Q \subset H,$$

where H is a Hilbert space. Suppose that Q has closed range. By the partial inverse of Q , we mean the bounded operator $M : H \rightarrow \operatorname{Dom} Q$ such that

$$\begin{aligned} QM + \pi &= I \text{ on } H, \\ MQ + \pi &= I \text{ on } \operatorname{Dom} Q, \end{aligned}$$

where $\pi : H \rightarrow \operatorname{Ker} Q$ is the orthogonal projection.

The main purpose of this work is to prove the following

Theorem 1.2. *With the notations and assumptions above,*

$$P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$$

is self-adjoint and P has L^2 closed range. Let $N : L^2(X) \rightarrow \text{Dom } P$ be the partial inverse and let $\Pi : L^2(X) \rightarrow \text{Ker } P$ be the orthogonal projection. Then,

$$(1.5) \quad \begin{aligned} \Pi, \tau, \tau_0 : H^s(X) &\rightarrow H^s(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ N : H^s(X) &\rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \end{aligned}$$

$$(1.6) \quad \Pi \equiv \tau \text{ on } X, \Pi \equiv \tau_0 \text{ on } X$$

and the kernel $\Pi(x, y) \in \mathcal{D}'(X \times X)$ of Π satisfies

$$(1.7) \quad \Pi(x, y) \equiv \int_0^\infty e^{i\varphi(x, y)t} a(x, y, t) dt + \int_0^\infty e^{-i\bar{\varphi}(x, y)t} \bar{a}(x, y, t) dt,$$

where

$$(1.8) \quad \begin{aligned} \varphi &\in C^\infty(X \times X), \quad \text{Im } \varphi(x, y) \geq 0, \quad d_x \varphi|_{x=y} = -\theta(x), \\ \varphi(x, y) &= -\bar{\varphi}(y, x), \\ \varphi(x, y) &= 0 \text{ if and only if } x = y, \end{aligned}$$

(see Theorem 1.8 and Theorem 1.10 for more properties of the phase φ), and

$$(1.9) \quad \begin{aligned} a(x, y, t) &\in S_{\text{cl}}^1(X \times X \times]0, \infty[), \\ a(x, y, t) &\sim \sum_{j=0}^\infty a_j(x, y) t^{1-j} \text{ in } S_{1,0}^1(X \times X \times]0, \infty[), \\ a_j(x, y) &\in C^\infty(X \times X), \quad j = 0, 1, \dots, \\ a_0(x, x) &= \frac{1}{2} \pi^{-n}, \quad \forall x \in X. \end{aligned}$$

(See section 2 and Definition 2.1 for the precise meanings of the notation \equiv and the Hörmander symbol spaces $S_{\text{cl}}^1(X \times X \times]0, \infty[)$ and $S_{1,0}^1(X \times X \times]0, \infty[)$).

Remark 1.3. With the notations and assumptions used in Theorem 1.2, it is easy to see that Π is real, that is $\Pi = \bar{\Pi}$.

Remark 1.4. With the notations and assumptions used in Theorem 1.2, let $S : L^2(X) \rightarrow \text{Ker } \bar{\partial}_b$ be the Szegő projection. That is, S is the orthogonal projection onto $\text{Ker } \bar{\partial}_b = \{u \in L^2(X); \bar{\partial}_b u = 0\}$ with respect to $(\cdot | \cdot)$. In view of the proof of Theorem 1.2 (see section 4), we see that $\Pi \equiv S + \bar{S}$ on X .

We have the classical formulas

$$(1.10) \quad \int_0^\infty e^{-tx} t^m dt = \begin{cases} m! x^{-m-1}, & \text{if } m \in \mathbb{Z}, \quad m \geq 0, \\ \frac{(-1)^m}{(-m-1)!} x^{-m-1} (\log x + c - \sum_1^{-m-1} \frac{1}{j}), & \text{if } m \in \mathbb{Z}, \quad m < 0. \end{cases}$$

Here $x \neq 0$, $\operatorname{Re} x \geq 0$ and c is the Euler constant, i.e. $c = \lim_{m \rightarrow \infty} (\sum_1^m \frac{1}{j} - \log m)$. Note that

$$(1.11) \quad \int_0^\infty e^{i\varphi(x,y)t} \sum_{j=0}^\infty a_j(x,y) t^{1-j} dt = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty e^{-t(-i(\varphi(x,y)+i\varepsilon))} \sum_{j=0}^\infty a_j(x,y) t^{1-j} dt.$$

We have the following corollary of Theorem 1.2

Corollary 1.5. *With the notations and assumptions used in Theorem 1.2, there exist $F_1, G_1 \in C^\infty(X \times X)$ such that*

$$\begin{aligned} \Pi(x, y) = & F_1(-i\varphi(x, y))^{-2} + G_1 \log(-i\varphi(x, y)) \\ & + \overline{F}_1(i\overline{\varphi}(x, y))^{-2} + \overline{G}_1 \log(i\overline{\varphi}(x, y)). \end{aligned}$$

Moreover, we have

$$(1.12) \quad \begin{aligned} F_1 = & a_0(x, y) + a_1(x, y)(-i\varphi(x, y)) + f_1(x, y)(-i\varphi(x, y))^2, \\ G_1 \equiv & \sum_0^\infty \frac{(-1)^{k+1}}{k!} a_{2+k}(x, y)(-i\varphi(x, y))^k, \end{aligned}$$

where $a_j(x, y)$, $j = 0, 1, \dots$, are as in (1.9) and $f_1(x, y) \in C^\infty(X \times X)$.

Put

$$\begin{aligned} \hat{\mathcal{P}}^\perp &:= \left\{ u \in L^2(X); (u | f) = 0, \forall f \in \hat{\mathcal{P}} \right\}, \\ \hat{\mathcal{P}}_0^\perp &:= \left\{ v \in L^2(X); (v | g) = 0, \forall g \in \hat{\mathcal{P}}_0 \right\}. \end{aligned}$$

From (1.6) and some standard argument in functional analysis (see section 4), we deduce

Corollary 1.6. *With the notations and assumptions above, we have*

$$\hat{\mathcal{P}}^\perp \cap \operatorname{Ker} P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \operatorname{Ker} P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}} \subset C^\infty(X)$$

and $\hat{\mathcal{P}}^\perp \cap \operatorname{Ker} P$, $\hat{\mathcal{P}}_0^\perp \cap \operatorname{Ker} P$, $\hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional.

We have the orthogonal decompositions

$$(1.13) \quad \begin{aligned} \operatorname{Ker} P &= \hat{\mathcal{P}}^\perp \oplus (\hat{\mathcal{P}}^\perp \cap \operatorname{Ker} P), \\ \operatorname{Ker} P &= \hat{\mathcal{P}}_0^\perp \oplus (\hat{\mathcal{P}}_0^\perp \cap \operatorname{Ker} P), \\ \hat{\mathcal{P}} &= \hat{\mathcal{P}}_0 \oplus (\hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}). \end{aligned}$$

From Corollary 1.6, we know that $\hat{\mathcal{P}}^\perp \cap \operatorname{Ker} P$, $\hat{\mathcal{P}}_0^\perp \cap \operatorname{Ker} P$, $\hat{\mathcal{P}}_0^\perp \cap \hat{\mathcal{P}}$ are all finite dimensional subsets of $C^\infty(X)$.

Since P is self-adjoint, $\operatorname{Spec} P \subset \mathbb{R}$. In section 5, we establish spectral theory for P .

Theorem 1.7. *With the notations and assumptions above, $\operatorname{Spec} P$ is a discrete subset in \mathbb{R} and for every $\lambda \in \operatorname{Spec} P$, $\lambda \neq 0$, λ is an eigenvalue of P and the eigenspace*

$$H_\lambda(P) := \{u \in \operatorname{Dom} P; Pu = \lambda u\}$$

is a finite dimensional subspace of $C^\infty(X)$.

1.1. The phase φ . In this section, we collect some properties of the phase function φ . We refer the reader to [7] and [8] for the proofs.

The following result describes the phase function φ in local coordinates.

Theorem 1.8. *With the assumptions and notations used in Theorem 1.2, for a given point $x_0 \in X$, let $\{Z_1\}$ be an orthonormal frame of $T^{1,0}X$ in a neighbourhood of x_0 , i.e. $\mathcal{L}_{x_0}(Z_1, \overline{Z}_1) = 1$. Take local coordinates $x = (x_1, x_2, x_3)$, $z = x_1 + ix_2$, defined on some neighbourhood of x_0 such that $\theta(x_0) = dx_3$, $x(x_0) = 0$, and for some $c \in \mathbb{C}$,*

$$Z_1 = \frac{\partial}{\partial z} - i\overline{z} \frac{\partial}{\partial x_3} - cx_3 \frac{\partial}{\partial x_3} + O(|x|^2).$$

Set $y = (y_1, y_2, y_3)$, $w = y_1 + iy_2$. Then, for φ in Theorem 1.2, we have

$$(1.14) \quad \text{Im } \varphi(x, y) \geq c \sum_{j=1}^2 |x_j - y_j|^2, \quad c > 0,$$

in some neighbourhood of $(0, 0)$ and

$$(1.15) \quad \begin{aligned} \varphi(x, y) = & -x_3 + y_3 + i|z - w|^2 \\ & + \left(i(\overline{z}w - z\overline{w}) + c(-zx_3 + wy_3) \right. \\ & \left. + \overline{c}(-\overline{z}x_3 + \overline{w}y_3) \right) + (x_3 - y_3)f(x, y) + O(|(x, y)|^3), \end{aligned}$$

where f is smooth and satisfies $f(0, 0) = 0$, $f(x, y) = \overline{f}(y, x)$.

Definition 1.9. With the assumptions and notations used in Theorem 1.2, let $\varphi_1(x, y), \varphi_2(x, y) \in C^\infty(X \times X)$. We assume that $\varphi_1(x, y)$ and $\varphi_2(x, y)$ satisfy (1.8) and (1.14). We say that $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are equivalent on X if for any $b_1(x, y, t) \in S_{\text{cl}}^1(X \times X \times]0, \infty[)$ we can find $b_2(x, y, t) \in S_{\text{cl}}^1(X \times X \times]0, \infty[)$ such that

$$\int_0^\infty e^{i\varphi_1(x, y)t} b_1(x, y, t) dt \equiv e^{i\varphi_2(x, y)t} b_2(x, y, t) dt \quad \text{on } X$$

and vice versa.

We characterize the phase φ

Theorem 1.10. *With the assumptions and notations used in Theorem 1.2, let $\varphi_1(x, y) \in C^\infty(X \times X)$. We assume that $\varphi_1(x, y)$ satisfies (1.8) and (1.14). $\varphi_1(x, y)$ and $\varphi(x, y)$ are equivalent on X in the sense of Definition 1.9 if and only if there is a function $h \in C^\infty(X \times X)$ such that $\varphi_1(x, y) - h(x, y)\varphi(x, y)$ vanishes to infinite order at $x = y$, for every $(x, x) \in X \times X$.*

2. PRELIMINARIES

We shall use the following notations: \mathbb{R} is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An element $\alpha = (\alpha_1, \dots, \alpha_n)$ of \mathbb{N}_0^n will be called a multiindex, the size of α is: $|\alpha| = \alpha_1 + \dots + \alpha_n$ and the length of α is $l(\alpha) = n$. For $m \in \mathbb{N}$, we write $\alpha \in \{1, \dots, m\}^n$ if $\alpha_j \in \{1, \dots, m\}$, $j = 1, \dots, n$. We say that α is strictly increasing if $\alpha_1 < \alpha_2 < \dots < \alpha_n$. We write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $x = (x_1, \dots, x_n)$, $\partial_x^\alpha =$

$\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$, $D_x = \frac{1}{i} \partial_x$, $D_{x_j} = \frac{1}{i} \partial_{x_j}$. Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, be coordinates of \mathbb{C}^n . We write $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $\bar{z}^\alpha = \bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}$, $\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n}$, $\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}})$, $j = 1, \dots, n$. $\frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha} = \partial_{\bar{z}}^\alpha = \partial_{\bar{z}_1}^{\alpha_1} \dots \partial_{\bar{z}_n}^{\alpha_n}$, $\partial_{\bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}})$, $j = 1, \dots, n$. For $j, s \in \mathbb{Z}$, set $\delta_{j,s} = 1$ if $j = s$, $\delta_{j,s} = 0$ if $j \neq s$.

Let M be a C^∞ paracompact manifold. We let TM and T^*M denote the tangent bundle of M and the cotangent bundle of M respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$ respectively. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between TM and T^*M . We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let E be a C^∞ vector bundle over M . The fiber of E at $x \in M$ will be denoted by E_x . Let F be another vector bundle over M . We write $E \boxtimes F$ to denote the vector bundle over $M \times M$ with fiber over $(x, y) \in M \times M$ consisting of the linear maps from E_x to F_y . Let $Y \subset M$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y, E)$ and $\mathcal{D}'(Y, E)$ respectively. Let $\mathcal{E}'(Y, E)$ be the subspace of $\mathcal{D}'(Y, E)$ whose elements have compact support in Y . For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y . Put

$$H_{\text{loc}}^m(Y, E) = \{u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \forall \varphi \in C_0^\infty(Y)\},$$

$$H_{\text{comp}}^m(Y, E) = H_{\text{loc}}^m(Y, E) \cap \mathcal{E}'(Y, E).$$

Let E and F be C^∞ vector bundles over a paracompact C^∞ manifold M equipped with a smooth density of integration. If $A : C_0^\infty(M, E) \rightarrow \mathcal{D}'(M, F)$ is continuous, we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of A . The following two statements are equivalent

- (a) A is continuous: $\mathcal{E}'(M, E) \rightarrow C^\infty(M, F)$,
- (b) $K_A \in C^\infty(M \times M, E_y \boxtimes F_x)$.

If A satisfies (a) or (b), we say that A is smoothing. Let $B : C_0^\infty(M, E) \rightarrow \mathcal{D}'(M, F)$ be a continuous operator. We write $A \equiv B$ (on M) if $A - B$ is a smoothing operator. We say that A is properly supported if $\text{Supp } K_A \subset M \times M$ is proper. That is, the two projections: $t_x : (x, y) \in \text{Supp } K_A \rightarrow x \in M$, $t_y : (x, y) \in \text{Supp } K_A \rightarrow y \in M$ are proper (i.e. the inverse images of t_x and t_y of all compact subsets of M are compact).

Let $H(x, y) \in \mathcal{D}'(M \times M, E_y \boxtimes F_x)$. We write H to denote the unique continuous operator $C_0^\infty(M, E) \rightarrow \mathcal{D}'(M, F)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

We recall Hörmander symbol spaces

Definition 2.1. Let $M \subset \mathbb{R}^N$ be an open set, $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$, $m \in \mathbb{R}$, $N_1 \in \mathbb{N}$. $S_{\rho, \delta}^m(M \times \mathbb{R}^{N_1})$ is the space of all $a \in C^\infty(M \times \mathbb{R}^{N_1})$ such that for all compact $K \Subset M$ and all $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0^{N_1}$, there is a constant $C > 0$ such that

$$\left| \partial_x^\alpha \partial_\theta^\beta a(z, \theta) \right| \leq C(1 + |\theta|)^{m - \rho|\beta| + \delta|\alpha|}, \quad (x, \theta) \in K \times \mathbb{R}^{N_1}.$$

We say that $S_{\rho,\delta}^m$ is the space of symbols of order m type (ρ, δ) . Put

$$S^{-\infty}(M \times \mathbb{R}^{N_1}) := \bigcap_{m \in \mathbb{R}} S_{\rho,\delta}^m(M \times \mathbb{R}^{N_1}).$$

Let $a_j \in S_{\rho,\delta}^{m_j}(M \times \mathbb{R}^{N_1})$, $j = 0, 1, 2, \dots$ with $m_j \rightarrow -\infty$, $j \rightarrow \infty$. Then there exists $a \in S_{\rho,\delta}^{m_0}(M \times \mathbb{R}^{N_1})$ unique modulo $S^{-\infty}(M \times \mathbb{R}^{N_1})$, such that $a - \sum_{j=0}^{k-1} a_j \in S_{\rho,\delta}^{m_k}(M \times \mathbb{R}^{N_1})$ for $k = 0, 1, 2, \dots$

If a and a_j have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j$ in $S_{\rho,\delta}^{m_0}(M \times \mathbb{R}^{N_1})$.

Let $S_{\text{cl}}^m(M \times \mathbb{R}^{N_1})$ be the space of all symbols $a(x, \theta) \in S_{1,0}^m(M \times \mathbb{R}^{N_1})$ with

$$a(x, \theta) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \theta) \text{ in } S_{1,0}^m(M \times \mathbb{R}^{N_1}),$$

with $a_k(x, \theta) \in C^\infty(M \times \mathbb{R}^{N_1})$ positively homogeneous of degree k in θ , that is, $a_k(x, \lambda\theta) = \lambda^k a_k(x, \theta)$, $\lambda \geq 1$, $|\theta| \geq 1$.

By using partition of unity, we extend the definitions above to the cases when M is a smooth paracompact manifold and when we replace $M \times \mathbb{R}^{N_1}$ by T^*M .

Let $\Omega \subset X$ be an open set. Let $a(x, \xi) \in S_{\frac{1}{2}, \frac{1}{2}}^k(T^*\Omega)$. We can define

$$A(x, y) = \frac{1}{(2\pi)^3} \int e^{i\langle x-y, \xi \rangle} a(x, \xi) d\xi$$

as an oscillatory integral and we can show that

$$A : C_0^\infty(\Omega) \rightarrow C^\infty(\Omega)$$

is continuous and has unique continuous extension:

$$A : \mathcal{E}'(\Omega) \rightarrow \mathcal{D}'(\Omega).$$

Definition 2.2. Let $k \in \mathbb{R}$. A pseudodifferential operator of order k type $(\frac{1}{2}, \frac{1}{2})$ is a continuous linear map $A : C_0^\infty(\Omega) \rightarrow \mathcal{D}'(\Omega)$ such that the distribution kernel of A is

$$A(x, y) = \frac{1}{(2\pi)^3} \int e^{i\langle x-y, \xi \rangle} a(x, \xi) d\xi$$

with $a \in S_{\frac{1}{2}, \frac{1}{2}}^k(T^*\Omega)$. We call $a(x, \xi)$ the symbol of A . We shall write $L_{\frac{1}{2}, \frac{1}{2}}^k(\Omega)$ to denote the space of pseudodifferential operators of order k type $(\frac{1}{2}, \frac{1}{2})$.

We recall the following classical result of Calderon-Vaillancourt (see chapter XVIII of Hörmander [6]).

Proposition 2.3. If $A \in L_{\frac{1}{2}, \frac{1}{2}}^k(\Omega)$. Then,

$$A : H_{\text{comp}}^s(\Omega) \rightarrow H_{\text{loc}}^{s-k}(\Omega)$$

is continuous, for all $s \in \mathbb{R}$. Moreover, if A is properly supported, then

$$A : H_{\text{loc}}^s(\Omega) \rightarrow H_{\text{loc}}^{s-k}(\Omega)$$

is continuous, for all $s \in \mathbb{R}$.

3. MICROLOCAL ANALYSIS FOR \square_b

We will reduce the analysis of the Paneitz operator to the analysis of Kohn Laplacian. We extend $\bar{\partial}_b$ to L^2 space by $\bar{\partial}_b : \text{Dom } \bar{\partial}_b \subset L^2(X) \rightarrow L^2_{(0,1)}(X)$, where $\text{Dom } \bar{\partial}_b := \{u \in L^2(X); \bar{\partial}_b u \in L^2_{(0,1)}(X)\}$. Let

$$\bar{\partial}_b^* : \text{Dom } \bar{\partial}_b^* \subset L^2_{(0,1)}(X) \rightarrow L^2(X)$$

be the L^2 adjoint of $\bar{\partial}_b$. The Gaffney extension of Kohn Laplacian is given by

$$(3.1) \quad \begin{aligned} \square_b &= \bar{\partial}_b^* \bar{\partial}_b : \text{Dom } \square_b \subset L^2(X) \rightarrow L^2(X), \\ \text{Dom } \square_b &:= \left\{ u \in L^2(X); u \in \text{Dom } \bar{\partial}_b, \bar{\partial}_b u \in \text{Dom } \bar{\partial}_b^* \right\}. \end{aligned}$$

It is well-known that \square_b is a positive self-adjoint operator. Moreover, the characteristic manifold of \square_b is given by

$$(3.2) \quad \Sigma = \{(x, \xi) \in T^*X; \xi = \lambda \theta(x), \lambda \neq 0\}.$$

Since X is embeddable, \square_b has L^2 closed range. Let $G : L^2(X) \rightarrow \text{Dom } \square_b$ be the partial inverse and let $S : L^2(X) \rightarrow \text{Ker } \square_b$ be the orthogonal projection (Szegő projection). Then,

$$(3.3) \quad \begin{aligned} \square_b G + S &= I \text{ on } L^2(X), \\ G \square_b + S &= I \text{ on } \text{Dom } \square_b. \end{aligned}$$

In [7], we proved that $G \in L^2_{\frac{1}{2}, \frac{1}{2}}(X)$, $S \in L^0_{\frac{1}{2}, \frac{1}{2}}(X)$ and we got explicit formulas of the kernels $G(x, y)$ and $S(x, y)$.

We introduce some notations. Let M be an open set in \mathbb{R}^N and let $f, g \in C^\infty(M)$. We write $f \asymp g$ if for every compact set $K \subset M$ there is a constant $c_K > 0$ such that $f \leq c_K g$ and $g \leq c_K f$ on K . Let $\Omega \subset X$ be an open set with real local coordinates $x = (x_1, x_2, x_3)$. We need

Definition 3.1. $a(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\Omega)$ is quasi-homogeneous of degree j if $a(t, x, \lambda \eta) = \lambda^j a(t, x, \eta)$ for all $\lambda > 0$.

We introduce some symbol classes

Definition 3.2. Let $\mu > 0$. We say that $a(t, x, \eta) \in \tilde{S}_\mu^m(\overline{\mathbb{R}}_+ \times T^*\Omega)$ if $a(t, x, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T^*\Omega)$ and there is a $a(x, \eta) \in S_{1,0}^m(T^*\Omega)$ such that for all indices $\alpha, \beta \in \mathbb{N}_0^3$, $\gamma \in \mathbb{N}_0$, every compact set $K \Subset \Omega$, there exists a constant $c_{\alpha, \beta, \gamma} > 0$ independent of t such that for all $t \in \overline{\mathbb{R}}_+$,

$$\left| \partial_t^\gamma \partial_x^\alpha \partial_\eta^\beta (a(t, x, \eta) - a(x, \eta)) \right| \leq c_{\alpha, \beta, \gamma} e^{-t\mu|\eta|} (1 + |\eta|)^{m+\gamma-|\beta|}, \quad x \in K, |\eta| \geq 1.$$

The following is well-known (see [7])

Theorem 3.3. *With the assumptions and notations above, $G \in L^2_{\frac{1}{2}, \frac{1}{2}}(X)$, $S \in L^0_{\frac{1}{2}, \frac{1}{2}}(X)$, $S(x, y) \equiv \int e^{i\varphi(x, y)t} a(x, y, t) dt$, where $\varphi(x, y) \in C^\infty(X \times X)$*

is as in (1.8) and

$$\begin{aligned} a(x, y, t) &\in S_{\text{cl}}^1(X \times X \times]0, \infty[), \\ a(x, y, t) &\sim \sum_{j=0}^{\infty} a_j(x, y) t^{1-j} \text{ in } S_{1,0}^1(X \times X \times]0, \infty[), \\ a_j(x, y) &\in C^\infty(X \times X), \quad j = 0, 1, \dots, \\ a_0(x, x) &= \frac{1}{2} \pi^{-n}, \quad \forall x \in X, \end{aligned}$$

and on every open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$, we have

(3.4)

$$G(x, y) \equiv \int_0^\infty \int_\Omega e^{i(\psi(t, x, \eta) - \langle y, \eta \rangle)} - t(i\psi'_t(t, x, \eta)a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta)) dt d\eta,$$

where $a(t, x, \eta) \in \tilde{S}_\mu^0(\overline{\mathbb{R}}_+ \times T^*\Omega)$, $\psi(t, x, \eta) \in \tilde{S}_\mu^1(\overline{\mathbb{R}}_+ \times T^*\Omega)$ for some $\mu > 0$, $\psi(t, x, \eta)$ is quasi-homogeneous of degree 1, $\psi(0, x, \eta) = \langle x, \eta \rangle$, $\text{Im } \psi \geq 0$ with equality precisely on $(\{0\} \times T^*\Omega \setminus 0) \cup (\mathbb{R}_+ \times \Sigma)$,

$$\psi(t, x, \eta) = \langle x, \eta \rangle \text{ on } \Sigma, \quad d_{x, \eta}(\psi - \langle x, \eta \rangle) = 0 \text{ on } \Sigma,$$

and

$$(3.5) \quad \text{Im } \psi(t, x, \eta) \asymp \left(|\eta| \frac{t |\eta|}{1 + t |\eta|} \right) \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2, \quad t \geq 0, \quad |\eta| \geq 1.$$

(See Theorem 3.4 below for the meaning of the integral (3.4).)

Proof. We only sketch the proof. For all the details, we refer the reader to Part I in [7]. We use the heat equation method. We work with some real local coordinates $x = (x_1, x_2, x_3)$ defined on Ω . We consider the problem

$$(3.6) \quad \begin{cases} (\partial_t + \square_b)u(t, x) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u(0, x) = v(x). \end{cases}$$

We look for an approximate solution of (3.6) of the form $u(t, x) = A(t)v(x)$,

$$(3.7) \quad A(t)v(x) = \frac{1}{(2\pi)^3} \int e^{i(\psi(t, x, \eta) - \langle y, \eta \rangle)} \alpha(t, x, \eta) v(y) dy d\eta$$

where formally

$$\alpha(t, x, \eta) \sim \sum_{j=0}^{\infty} \alpha_j(t, x, \eta),$$

with $\alpha_j(t, x, \eta)$ quasi-homogeneous of degree $-j$.

The full symbol of \square_b equals $\sum_{j=0}^2 p_j(x, \xi)$, where $p_j(x, \xi)$ is positively homogeneous of order $2 - j$ in the sense that

$$p_j(x, \lambda \eta) = \lambda^{2-j} p_j(x, \eta), \quad |\eta| \geq 1, \quad \lambda \geq 1.$$

We apply $\partial_t + \square_b$ formally inside the integral in (3.7) and then introduce the asymptotic expansion of $\square_b(\alpha e^{i\psi})$. Set $(\partial_t + \square_b)(\alpha e^{i\psi}) \sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase $\psi(t, x, \eta)$ should solve

$$(3.8) \quad \begin{cases} \frac{\partial \psi}{\partial t} - ip_0(x, \psi'_x) = O(|\text{Im } \psi|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = \langle x, \eta \rangle. \end{cases}$$

This equation can be solved with $\text{Im } \psi(t, x, \eta) \geq 0$ and the phase $\psi(t, x, \eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\begin{aligned} \psi(t, x, \eta) &= \langle x, \eta \rangle \text{ on } \Sigma, \quad d_{x, \eta}(\psi - \langle x, \eta \rangle) = 0 \text{ on } \Sigma, \\ \text{Im } \psi(t, x, \eta) &\asymp \left(|\eta| \frac{t |\eta|}{1 + t |\eta|} \right) \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2, \quad |\eta| \geq 1. \end{aligned}$$

Furthermore, there exists $\psi(\infty, x, \eta) \in C^\infty(\Omega \times \mathbb{R}^3)$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset \Omega \times \mathbb{R}^3$ there is a constant $c_K > 0$ such that

$$\text{Im } \psi(\infty, x, \eta) \geq c_K |\eta| \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2, \quad |\eta| \geq 1.$$

If $\lambda \in C(T^*\Omega \setminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min \lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of \square_b , then the solution $\psi(t, x, \eta)$ of (3.8) can be chosen so that for every compact set $K \subset \Omega \times \mathbb{R}^3$ and all indices α, β, γ , there is a constant $c_{\alpha, \beta, \gamma, K}$ such that

$$\left| \partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma (\psi(t, x, \eta) - \psi(\infty, x, \eta)) \right| \leq c_{\alpha, \beta, \gamma, K} e^{-\lambda(x, \eta)t} \text{ on } \overline{\mathbb{R}}_+ \times K.$$

We obtain the transport equations

$$(3.9) \quad \begin{cases} T(t, x, \eta, \partial_t, \partial_x) \alpha_0 = O(|\text{Im } \psi|^N), \quad \forall N, \\ T(t, x, \eta, \partial_t, \partial_x) \alpha_j + l_j(t, x, \eta, \alpha_0, \dots, \alpha_{j-1}) = O(|\text{Im } \psi|^N), \quad \forall N, \quad j \in \mathbb{N}. \end{cases}$$

It was proved in [7] that (3.9) can be solved. Moreover, there exist positively homogeneous functions of degree $-j$

$$\alpha_j(\infty, x, \eta) \in C^\infty(T^*\Omega), \quad j = 0, 1, 2, \dots,$$

such that $\alpha_j(t, x, \eta)$ converges exponentially fast to $\alpha_j(\infty, x, \eta)$, $t \rightarrow \infty$, for all $j \in \mathbb{N}_0$. Set

$$\tilde{G} = \frac{1}{(2\pi)^3} \int \int_0^\infty e^{i(\psi(t, x, \eta) - \langle y, \eta \rangle) - t} (i\psi'_t(t, x, \eta) \alpha(t, x, \eta) + \frac{\partial \alpha}{\partial t}(t, x, \eta)) dt d\eta$$

and

$$\tilde{S} = \frac{1}{(2\pi)^3} \int e^{i(\psi(\infty, x, \eta) - \langle y, \eta \rangle)} \alpha(\infty, x, \eta) d\eta.$$

We can show that \tilde{G} is a pseudodifferential operator of order -1 type $(\frac{1}{2}, \frac{1}{2})$, \tilde{S} is a pseudodifferential operator of order 0 type $(\frac{1}{2}, \frac{1}{2})$ satisfying

$$\tilde{S} + \square_b \tilde{G} \equiv I, \quad \square_b \tilde{S} \equiv 0.$$

Moreover, from global theory of complex Fourier integral operators, we can show that $\tilde{S} \equiv \int e^{i\varphi(x, y)t} a(x, y, t) dt$. Furthermore, by using some standard argument in functional analysis, we can show that $\tilde{G} \equiv G$, $\tilde{S} \equiv S$. \square

Until further notice, we work in an open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$. The following is well-known (see Chapter 5 in [7])

Theorem 3.4. *With the notations and assumptions used in Theorem 3.3, let $\chi \in C_0^\infty(\mathbb{R}^3)$ be equal to 1 near the origin. Put*

$$G_\varepsilon(x, y) = \int_0^\infty \int_0^\infty e^{i(\psi(t, x, \eta) - \langle y, \eta \rangle)} - t(i\psi'_t(t, x, \eta)a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta))\chi(\varepsilon\eta)dt d\eta,$$

where $\psi(t, x, \eta)$, $a(t, x, \eta)$ are as in (3.4). For $u \in C_0^\infty(\Omega)$, we can show that

$$Gu := \lim_{\varepsilon \rightarrow 0} \int G_\varepsilon(x, y)u(y)dy \in C^\infty(\Omega)$$

and

$$G : C_0^\infty(\Omega) \rightarrow C^\infty(\Omega),$$

$$u \rightarrow \lim_{\varepsilon \rightarrow 0} \int G_\varepsilon(x, y)u(y)dy$$

is continuous.

Moreover, $G \in L_{\frac{1}{2}, \frac{1}{2}}^{-1}(\Omega)$ with symbol

$$\int_0^\infty e^{i(\psi(t, x, \eta) - \langle x, \eta \rangle)} - t(i\psi'_t(t, x, \eta)a(t, x, \eta) + \frac{\partial a}{\partial t}(t, x, \eta))dt \in S_{\frac{1}{2}, \frac{1}{2}}^{-1}(T^*\Omega).$$

We need the following (see Lemma 5.13 in [7] for a proof)

Lemma 3.5. *With the notations and assumptions used in Theorem 3.3, for every compact set $K \subset \Omega$ and all $\alpha \in \mathbb{N}_0^3$, $\beta \in \mathbb{N}_0^3$, there exists a constant $c_{\alpha, \beta, K} > 0$ such that*

(3.10)

$$\begin{aligned} & \left| \partial_x^\alpha \partial_\eta^\beta (e^{i(\psi(t, x, \eta) - \langle x, \eta \rangle)} t\psi'_t(t, x, \eta)) \right| \\ & \leq c_{\alpha, \beta, K} (1 + |\eta|)^{\frac{|\alpha| - |\beta|}{2}} e^{-t\mu|\eta|} e^{-\text{Im} \psi(t, x, \eta)} (1 + \text{Im} \psi(t, x, \eta))^{1 + \frac{|\alpha| + |\beta|}{2}}, \end{aligned}$$

where $x \in K$, $t \in \overline{\mathbb{R}}_+$, $|\eta| \geq 1$ and $\mu > 0$ is a constant independent of α , β and K .

In this work, we need

Theorem 3.6. *Let $L \in C^\infty(X, T^{1,0}X \oplus T^{0,1}X)$. Then, $L \circ G \in L_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}}(X)$.*

Proof. We work on an open local coordinate patch $\Omega \subset X$ with real local coordinates $x = (x_1, x_2, x_3)$. Let $l(x, \eta) \in C^\infty(T^*\Omega)$ be the symbol of L . Then, $l(x, \lambda\eta) = \lambda l(x, \eta)$, $\lambda > 0$. It is well-known (see Chapter 5 in [7]) that

$$(LG)(x, y) \equiv \int e^{i\langle x - y, \eta \rangle} \alpha(x, \eta) d\eta,$$

where

$$\alpha(x, \eta) = \alpha_0(x, \eta) + \alpha_1(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^0(T^*\Omega),$$

$$\alpha_0(x, \eta) = \int e^{i(\psi(t, x, \eta) - \langle x, \eta \rangle)} (-1) l(x, \psi'_x(t, x, \eta)) t\psi'_t(t, x, \eta) a(t, x, \eta) dt,$$

$$\alpha_1(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^{-1}(T^*\Omega).$$

Here $a(t, x, \eta) \in \tilde{S}_\mu^0(\overline{\mathbb{R}}_+ \times T^*\Omega)$, $\mu > 0$. We only need to prove that $\alpha_0(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}}(T^*\Omega)$. Fix $\alpha, \beta \in \mathbb{N}_0^3$. From (3.10), (3.5) and notice that $l(x, \psi'_x(t, x, \eta)) = 0$ at Σ , we can check that

$$\begin{aligned}
(3.11) \quad & \left| \partial_x^\alpha \partial_\eta^\beta \alpha_0(x, \eta) \right| \\
& \leq \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int \left| \partial_x^{\alpha'} \partial_\eta^{\beta'} (e^{i(\psi(t, x, \eta) - \langle x, \eta \rangle)} t \psi'_t(t, x, \eta)) \right| \times \\
& \quad \left| \partial_x^{\alpha''} \partial_\eta^{\beta''} (l(x, \psi'_x(t, x, \eta)) a(t, x, \eta)) \right| dt \\
& \leq C_{\alpha, \beta} \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-t\mu|\eta|} e^{-\frac{1}{2}\text{Im} \psi(t, x, \eta)} \times \\
& \quad (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^{\max\{0, 1 - |\beta''|\}} dt \\
& \leq \tilde{C}_{\alpha, \beta} \sum_{|\alpha'| + |\alpha''| = |\alpha|, |\beta'| + |\beta''| = |\beta|} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-c \frac{t|\eta|^2}{1+t|\eta|}} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2 \times \\
& \quad e^{-t\mu|\eta|} (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^{\max\{0, 1 - |\beta''|\}} dt,
\end{aligned}$$

where $c > 0$, $\mu > 0$, $C_{\alpha, \beta} > 0$ and $\tilde{C}_{\alpha, \beta} > 0$ are constants.

When $|\beta''| = 0$, we have

$$\begin{aligned}
(3.12) \quad & \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-c \frac{t|\eta|^2}{1+t|\eta|}} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2 \times \\
& \quad e^{-t\mu|\eta|} (1 + |\eta|)^{1 - |\beta''|} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right) dt \\
& \leq \tilde{c} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \\
& \leq \tilde{c}_1 \int_0^{\frac{1}{1+|\eta|}} (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \\
& \quad + \tilde{c}_2 \int_{\frac{1}{1+|\eta|}}^\infty (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} \frac{1}{\sqrt{t}} (1 + |\eta|)^{-|\beta''|} e^{-\frac{1}{2}t\mu|\eta|} dt \\
& \leq \tilde{c}_3 (1 + |\eta|)^{-\frac{1}{2} - |\beta''| + \frac{|\alpha'| - |\beta'|}{2}},
\end{aligned}$$

where $|\eta| \geq 1$, $\tilde{c}_1 > 0$, $\tilde{c}_2 > 0$ and $\tilde{c}_3 > 0$ are constants.

When $|\beta''| \geq 1$, we have

$$\begin{aligned}
(3.13) \quad & \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} e^{-c \frac{t|\eta|^2}{1+t|\eta|}} \left(\text{dist} \left((x, \frac{\eta}{|\eta|}), \Sigma \right) \right)^2 \times \\
& e^{-t\mu|\eta|} (1 + |\eta|)^{1-|\beta''|} dt \\
& \leq \hat{c} \int (1 + |\eta|)^{\frac{|\alpha'| - |\beta'|}{2}} (1 + |\eta|)^{1-|\beta''|} e^{-t\mu|\eta|} dt \\
& \leq \hat{c}_1 (1 + |\eta|)^{-|\beta''| + \frac{|\alpha'| - |\beta'|}{2}} \\
& \leq \hat{c}_2 (1 + |\eta|)^{-\frac{1}{2} + \frac{|\alpha'| - |\beta'| - |\beta''|}{2}},
\end{aligned}$$

where $|\eta| \geq 1$, $\hat{c}_1 > 0$, $\hat{c}_2 > 0$ are constants.

From (3.11), (3.12) and (3.13), we conclude that $\alpha_0(x, \eta) \in S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}}(T^*\Omega)$. The theorem follows. \square

4. MICROLOCAL HODGE DECOMPOSITION THEOREMS FOR P AND THE PROOF OF THEOREM 1.2

By using Theorem 3.3 and Theorem 3.6, we will establish microlocal Hodge decomposition theorems for P in this section. Let $G \in L_{\frac{1}{2}, \frac{1}{2}}^{-1}(X)$, $S \in L_{\frac{1}{2}, \frac{1}{2}}^0(X)$ be as in Theorem 3.3. From (1.2) and (3.3), we have

$$\begin{aligned}
(4.1) \quad & P \overline{G} G = (\square_b \overline{\square}_b + L_1 \circ L_2 + L_3) \overline{G} G \\
& = \square_b (I - \overline{S}) G + L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G \\
& = I - S - \square_b \overline{S} G + L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G \\
& = I - S - \overline{S} \square_b G + \overline{S} \square_b G - \square_b \overline{S} G + L_1 \circ L_2 \circ \overline{G} G + L_3 \overline{G} G \\
& = I - S - \overline{S} (I - S) + [\overline{S}, \square_b] G + L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G \\
& = I - S - \overline{S} + \overline{S} S + [\overline{S}, \square_b] G + L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G.
\end{aligned}$$

We need

Lemma 4.1. *We have*

$$\begin{aligned}
(4.2) \quad & [\overline{S}, \square_b] G + L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G \\
& : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, for every } s \in \mathbb{Z}.
\end{aligned}$$

Proof. From Theorem 3.6, we see that $L_1 \circ L_2 \overline{G} \in L_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(X)$. Thus,

$$(4.3) \quad L_1 \circ L_2 \overline{G} G + L_3 \overline{G} G : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

Since $\square_b \overline{S} = \overline{S} \square_b = 0$, we have

$$(4.4) \quad [\overline{S}, \square_b] = [\overline{S}, \square_b - \overline{\square}_b].$$

Since the principal symbol of \square_b is real, $\square_b - \overline{\square}_b$ is a first order partial differential operator. From this observation and note that $\overline{S} \in L_{\frac{1}{2}, \frac{1}{2}}^0(X)$, it

is not difficult to see that $[\overline{S}, \square_b - \overline{\square}_b] \in L^{\frac{1}{2}, \frac{1}{2}}(X)$. From this and (4.4), we conclude that

$$(4.5) \quad [\overline{S}, \square_b]G : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

From (4.5) and (4.3), (4.2) follows. \square

We also need

Lemma 4.2. *We have $\overline{S}S \equiv 0$ on X , $S\overline{S} \equiv 0$ on X .*

Proof. We first notice that $\overline{S} \circ S$ is smoothing away $x = y$. We have

$$(4.6) \quad \begin{aligned} \overline{S} \circ S(x, y) &\equiv \int_{\sigma>0, t>0} e^{-i\overline{\varphi}(x, w)\sigma + i\varphi(w, y)t} \overline{a}(x, w, \sigma) a(w, y, t) d\sigma dv_X(w) dt \\ &\equiv \int_{s>0, t>0} e^{it(-\overline{\varphi}(x, w)s + \varphi(w, y))} t \overline{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \end{aligned}$$

where $dv_X = \theta \wedge d\theta$ is the volume form. Take $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ with $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\chi = 0$ on $]-\infty, -1] \cup [1, \infty[$. From (4.6), we have

$$(4.7) \quad \begin{aligned} \overline{S} \circ S(x, y) &\equiv I_\varepsilon + II_\varepsilon, \\ I_\varepsilon &= \int_{s>0, t>0} e^{it(-\overline{\varphi}(x, w)s + \varphi(w, y))} \chi\left(\frac{|x-w|^2}{\varepsilon}\right) t \overline{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \\ II_\varepsilon &= \int_{s>0, t>0} e^{it(-\overline{\varphi}(x, w)s + \varphi(w, y))} \left(1 - \chi\left(\frac{|x-w|^2}{\varepsilon}\right)\right) \\ &\quad \times t \overline{a}(x, w, st) a(w, y, t) ds dv_X(w) dt, \end{aligned}$$

where $\varepsilon > 0$ is a small constant. Since $\varphi(x, w) = 0$ if and only if $x = w$, we can integrate by parts with respect to s and conclude that II_ε is smoothing. Since $\overline{S} \circ S$ is smoothing away $x = y$, we may assume that $|x - y| < \varepsilon$. Since $d_w(-\overline{\varphi}(x, w)s + \varphi(w, y))|_{x=y=w} = -\omega_0(x)(s+1) \neq 0$, if $\varepsilon > 0$ is small, we can integrate by parts with respect to w and conclude that I_ε is smoothing. We get $\overline{S} \circ S \equiv 0$ on X . Similarly, we can repeat the procedure above and conclude that $S \circ \overline{S} \equiv 0$ on X . The lemma follows. \square

Put

$$(4.8) \quad R_0 = \overline{S}S + [\overline{S}, \square_b]G + L_1 \circ L_2 \overline{G}G + L_3 \overline{G}G.$$

From Lemma 4.1 and Lemma 4.2, we see that

$$(4.9) \quad R_0 : H^s(X) \rightarrow H^{s+\frac{1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

We can prove

Theorem 4.3. *With the assumptions and notations above, for every $m \in \mathbb{N}_0$, there are continuous operators*

$$R_m, A_m : C_0^\infty(X) \rightarrow \mathcal{D}'(X)$$

such that

$$\begin{aligned}
 & \text{P } A_m + S + \overline{S} = I + R_m, \\
 (4.10) \quad & A_m : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\
 & R_m : H^s(X) \rightarrow H^{s+\frac{m+1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.
 \end{aligned}$$

Proof. From (4.8) and (4.1), we have

$$(4.11) \quad \text{P } \overline{G}G + S + \overline{S} = I + R_0.$$

Since $(S + \overline{S})\text{P} = 0$, from (4.11), we have

$$(4.12) \quad (S + \overline{S})^2 = S + \overline{S} + (S + \overline{S})R_0.$$

From Lemma 4.2, we have

$$(4.13) \quad (S + \overline{S})^2 = S^2 + S\overline{S} + \overline{S}S + \overline{S}^2 \equiv S + \overline{S}.$$

From (4.12) and (4.13), we conclude that

$$(4.14) \quad (S + \overline{S})R_0 \equiv 0 \text{ on } X.$$

Fix $m \in \mathbb{N}_0$. From (4.11), we have

$$\begin{aligned}
 & \text{P } \overline{G}G(I - R_0 + R_0^2 + \cdots + (-R_0)^m) \\
 (4.15) \quad & + (S + \overline{S})(I - R_0 + R_0^2 + \cdots + (-R_0)^m) \\
 & = (I + R_0)(I - R_0 + R_0^2 + \cdots + (-R_0)^m) = I + R_0(-R_0)^m.
 \end{aligned}$$

From (4.14), we have

$$(4.16) \quad (S + \overline{S})(I - R_0 + R_0^2 + \cdots + (-R_0)^m) = S + \overline{S} - F, \quad F \text{ is smoothing.}$$

Put $A_m = \overline{G}G(I - R_0 + R_0^2 + \cdots + (-R_0)^m)$, $R_m = R_0(-R_0)^m + F$. From (4.15), (4.16) and (4.9), we obtain

$$\begin{aligned}
 & \text{P } A_m + S + \overline{S} = I + R_m, \\
 & A_m : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\
 & R_m : H^s(X) \rightarrow H^{s+\frac{m+1}{2}}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.
 \end{aligned}$$

The theorem follows. \square

Lemma 4.4. *Let $u \in \text{Dom P}$. Then, $u = u^0 + (S + \overline{S})u$, for some $u^0 \in H^2(X) \cap \text{Dom P}$.*

Proof. Fix $m \geq 3$, $m \in \mathbb{N}$. Let A_m, R_m be as in (4.10) and let A_m^* and R_m^* be the adjoints of A_m and R_m with respect to $(\cdot | \cdot)$ respectively. Then,

$$(4.17) \quad A_m^* \text{P} + S + \overline{S} = I + R_m^*.$$

Let $u \in \text{Dom P}$. Then, $\text{P } u = v \in L^2(X)$. From (4.17), it is easy to see that

$$u = A_m^* v - R_m^* u + (S + \overline{S})u.$$

Since $A_m^* v - R_m^* u \in H^2(X)$ and $(S + \overline{S})u \in \text{Ker P} \subset \text{Dom P}$, the lemma follows. \square

Lemma 4.5. *Let $u \in \text{Dom P}$. Then,*

$$((S + \overline{S})u | \text{P } g) = 0, \quad \forall g \in \text{Dom P}.$$

Proof. Let $u, g \in \text{Dom } P$. Take $u_j, g_j \in C^\infty(X)$, $j = 1, 2, \dots$, $u_j \rightarrow u \in L^2(X)$ as $j \rightarrow \infty$ and $g_j \rightarrow g \in L^2(X)$ as $j \rightarrow \infty$. Then, $(S + \bar{S})u_j \rightarrow (S + \bar{S})u$ in $L^2(X)$ as $j \rightarrow \infty$ and $P g_j \rightarrow P g$ in $H^{-4}(X)$ as $j \rightarrow \infty$. Thus,

$$(4.18) \quad ((S + \bar{S})u | P g) = \lim_{j \rightarrow \infty} ((S + \bar{S})u_j | P g) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} ((S + \bar{S})u_j | P g_k).$$

For any j, k , $((S + \bar{S})u_j | P g_k) = (P(S + \bar{S})u_j | g_k) = 0$. From this observation and (4.18), the lemma follows. \square

Now, we can prove

Theorem 4.6. *The operator $P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$ is self-adjoint.*

Proof. Let $u, v \in \text{Dom } P$. From Lemma 4.4, we have

$$\begin{aligned} u &= u^0 + (S + \bar{S})u, \quad u^0 \in H^2(X) \cap \text{Dom } P, \\ v &= v^0 + (S + \bar{S})v, \quad v^0 \in H^2(X) \cap \text{Dom } P. \end{aligned}$$

From Lemma 4.5, we see that

$$(4.19) \quad (u | P v) = (u^0 | P v^0), \quad (P u | v) = (P u^0 | v^0).$$

Let $g_j, f_j \in C^\infty(X)$, $j = 1, 2, \dots$, $g_j \rightarrow u^0 \in H^2(X)$ as $j \rightarrow \infty$ and $f_j \rightarrow v^0 \in H^2(X)$ as $j \rightarrow \infty$. We have

$$(4.20) \quad \begin{aligned} (g_j | P f_j) &= (g_j - u^0 | P f_j) + (u^0 | P f_j) \\ &= (g_j - u^0 | P f_j) + (u^0 | P v^0) + (u^0 | P(f_j - v^0)). \end{aligned}$$

Now,

$$(4.21) \quad \begin{aligned} |(g_j - u^0 | P f_j)| &\leq C_0 \|g_j - u^0\|_2 \|P f_j\|_{-2} \\ &\leq C_1 \|g_j - u^0\|_2 \|f_j\|_2 \rightarrow 0 \text{ as } j \rightarrow \infty \end{aligned}$$

and

$$(4.22) \quad \begin{aligned} |(u^0 | P(f_j - v^0))| &\leq C_2 \|u^0\|_2 \|P(f_j - v^0)\|_{-2} \\ &\leq C_3 \|u^0\|_2 \|f_j - v^0\|_2 \rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned}$$

where $C_0 > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ are constants and $\|\cdot\|_s$ denotes the standard Sobolev norm of order s on X . From (4.20), (4.21) and (4.22), we obtain

$$(4.23) \quad (u^0 | P v^0) = \lim_{j \rightarrow \infty} (g_j | P f_j).$$

For each j , it is clearly that $(g_j | P f_j) = (P g_j | f_j)$. We can repeat the procedure above and conclude that

$$\lim_{j \rightarrow \infty} (P g_j | f_j) = (P u^0 | v^0).$$

From this observation, (4.23) and (4.19), we conclude that

$$(4.24) \quad (P u | v) = (u | P v), \quad \forall u, v \in \text{Dom } P.$$

Let $P^* : \text{Dom } P^* \subset L^2(X) \rightarrow L^2(X)$ be the Hilbert space adjoint of P . From (4.24), we deduce that $\text{Dom } P \subset \text{Dom } P^*$ and $P u = P^* u$, $\forall u \in \text{Dom } P^*$.

Let $v \in \text{Dom } P^*$. By definition, there is a $f \in L^2(X)$ such that

$$(v | P g) = (f | g), \quad \forall g \in \text{Dom } P.$$

Since $C^\infty(X) \subset \text{Dom } P$, $P v = f$ in the sense of distribution. Since $f \in L^2(X)$, $v \in \text{Dom } P$ and $P v = P^* v = f$. The theorem follows. \square

Theorem 4.7. *The operator $P : \text{Dom } P \subset L^2(X) \rightarrow L^2(X)$ has closed range.*

Proof. Fix $m \in \mathbb{N}_0$, let A_m, R_m be as in (4.10) and let A_m^* and R_m^* be the adjoints of A_m and R_m with respect to $(\cdot | \cdot)$ respectively. Then,

$$(4.25) \quad A_m^* P + S + \overline{S} = I + R_m^*.$$

Now, we claim that there is a constant $C > 0$ such that

$$(4.26) \quad \|P u\| \geq C \|u\|, \quad \forall u \in \text{Dom } P \cap (\text{Ker } P)^\perp.$$

If the claim is not true, then we can find $u_j \in \text{Dom } P \cap (\text{Ker } P)^\perp$ with $\|u_j\| = 1$, $j = 1, 2, \dots$, such that

$$(4.27) \quad \|P u_j\| \leq \frac{1}{j} \|u_j\|, \quad j = 1, 2, \dots$$

From (4.25), we have

$$(4.28) \quad u_j = A_m^* P u_j + (S + \overline{S}) u_j - R_m u_j, \quad j = 1, 2, \dots$$

Since $u_j \in (\text{Ker } P)^\perp$, $j = 1, 2, \dots$, and $P(S + \overline{S}) = 0$, we have

$$(4.29) \quad (S + \overline{S}) u_j = 0, \quad j = 1, 2, \dots$$

From (4.28) and (4.29), we get

$$(4.30) \quad u_j = A_m^* P u_j - R_m u_j, \quad j = 1, 2, \dots$$

From (4.30) and Rellich's theorem, we can find subsequence $\{u_{j_s}\}_{s=1}^\infty$, $1 \leq j_1 < j_2 < \dots$, $u_{j_s} \rightarrow u$ in $L^2(X)$. From (4.27), we see that $P u = 0$. Hence, $u \in \text{Ker } P$. Since $u_j \in (\text{Ker } P)^\perp$, $j = 1, 2, \dots$, we get a contradiction. The claim (4.26) follows. From (4.26), the theorem follows. \square

In view of Theorem 4.6 and Theorem 4.7, we know that P is self-adjoint and P has closed range. Let $N : L^2(X) \rightarrow \text{Dom } P$ be the partial inverse and let $\Pi : L^2(X) \rightarrow \text{Ker } P$ be the orthogonal projection. We can prove

Theorem 4.8. *With the notations and assumptions above, we have*

$$(4.31) \quad \begin{aligned} \Pi : H^s(X) &\rightarrow H^s(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ N : H^s(X) &\rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}, \\ \Pi &\equiv S + \overline{S}. \end{aligned}$$

Proof. Fix $m \in \mathbb{N}_0$. Let A_m, R_m be as in Theorem 4.3. Then,

$$P A_m + S + \overline{S} = I + R_m.$$

Thus,

$$(4.32) \quad \Pi + \Pi R_m = \Pi(P A_m + S + \overline{S}) = \Pi(S + \overline{S}) = S + \overline{S}.$$

From (4.10) and (4.32), we have

$$(4.33) \quad \Pi - (S + \overline{S}) : H^{-\frac{m+1}{2}}(X) \rightarrow L^2(X) \text{ is continuous.}$$

By taking adjoint in (4.33), we get

$$(4.34) \quad \Pi - (S + \overline{S}) : L^2(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.}$$

From (4.33) and (4.34), we have

$$(4.35) \quad (\Pi - (S + \overline{S}))^2 : H^{-\frac{m+1}{2}}(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.}$$

Now,

$$(4.36) \quad \begin{aligned} (\Pi - (S + \overline{S}))^2 &= \Pi - \Pi(S + \overline{S}) - (S + \overline{S})\Pi + (S + \overline{S})^2 \\ &= \Pi - (S + \overline{S}) - (S + \overline{S}) + S + \overline{S} + S\overline{S} + \overline{S}S \\ &\equiv \Pi - (S + \overline{S}) \quad (\text{here we used Lemma 4.2}). \end{aligned}$$

From (4.35) and (4.36), we conclude that

$$\Pi - (S + \overline{S}) : H^{-\frac{m+1}{2}}(X) \rightarrow H^{\frac{m+1}{2}}(X) \text{ is continuous.}$$

Since m is arbitrary, we get

$$(4.37) \quad \Pi \equiv S + \overline{S}.$$

Now,

$$(4.38) \quad N(P A_m + S + \overline{S}) = N(I + R_m).$$

Note that $NP = I - \Pi$, $N\Pi = 0$. From this observation, we have

$$(4.39) \quad N(P A_m + S + \overline{S}) = (I - \Pi)A_m + NF,$$

where $F \equiv 0$ (here we used (4.37)). From (4.39) and (4.38), we have

$$(4.40) \quad N - A_m = -\Pi A_m + NF - NR_m.$$

From (4.37) and (4.40), we have

$$(4.41) \quad \begin{aligned} N - A_m^* &= -A_m^* \Pi + F^* N - R_m^* N \\ &= -A_m^* \Pi + F^* (-\Pi A_m + NF - NR_m + A_m) \\ &\quad - R_m^* (-\Pi A_m + NF - NR_m + A_m) \\ &: H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall -\frac{m+1}{2} \leq s \leq \frac{m-3}{2}, s \in \mathbb{Z}, \end{aligned}$$

where A_m^* , F^* , R_m^* are adjoints of A_m , F , R_m respectively. Note that

$$A_m^* : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

From this observation, (4.41) and note that m is arbitrary, we conclude that

$$N : H^s(X) \rightarrow H^{s+2}(X) \text{ is continuous, } \forall s \in \mathbb{Z}.$$

The theorem follows. \square

Let τ and τ_0 be as in (1.4). Now, we can prove

Theorem 4.9. *We have $\tau \equiv \Pi$ on X , $\tau_0 \equiv \Pi$ on X .*

Proof. Since $\hat{\mathcal{P}} \subset \text{Ker } P$, we have $\Pi\tau = \tau$. From this observation and (4.31), we get

$$(4.42) \quad (S + \bar{S})\tau - \tau = F\tau,$$

where F is a smoothing operator. It is clearly that $(S + \bar{S})\tau = \tau(S + \bar{S}) = S + \bar{S}$. From this observation and (4.42), we get $S + \bar{S} - \tau = F\tau$ and hence $S + \bar{S} - \tau = \tau F^*$, where F^* is the adjoint of F . Thus,

$$(4.43) \quad (S + \bar{S} - \tau)(S + \bar{S} - \tau) = F\tau^2 F^* \equiv 0.$$

Now,

$$(4.44) \quad \begin{aligned} (S + \bar{S} - \tau)^2 &= (S + \bar{S})^2 - (S + \bar{S})\tau - \tau(S + \bar{S}) + \tau^2 \\ &= S + S\bar{S} + \bar{S}S + \bar{S} - S - \bar{S} - S - \bar{S} + \tau \\ &\equiv \tau - (S + \bar{S}) \quad (\text{here we used Lemma 4.2}). \end{aligned}$$

From (4.44), (4.43) and (4.31), we get $\tau \equiv \Pi$.

Similarly, we can repeat the procedure above and conclude that $\tau_0 \equiv \Pi$. The theorem follows. \square

From Theorem 4.6, Theorem 4.7, Theorem 4.8, Theorem 4.9 and Theorem 3.3, we get Theorem 1.2.

Corollary 4.10. *We have*

$$\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \bigcap \text{Ker } P \subset C^\infty(X), \quad \hat{\mathcal{P}}_0^\perp \bigcap \hat{\mathcal{P}} \subset C^\infty(X)$$

and $\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \bigcap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \bigcap \hat{\mathcal{P}}$ are all finite dimensional.

Proof. If $\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P$ is infinite dimensional, then we can find

$$f_j \in \hat{\mathcal{P}}^\perp \bigcap \text{Ker } P, \quad j = 1, 2, \dots,$$

such that $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$. Since $f_j \in \text{Ker } P$, $j = 1, 2, \dots$, $f_j = \Pi f_j$, $j = 1, 2, \dots$. From Theorem 4.9, we have

$$(4.45) \quad f_j = \tau f_j + F f_j, \quad j = 1, 2, 3, \dots,$$

where F is a smoothing operator. Since $f_j \in \hat{\mathcal{P}}^\perp$, $j = 1, 2, \dots$, $\tau f_j = 0$, $j = 1, 2, \dots$. From this observation and (4.45), we get

$$(4.46) \quad f_j = F f_j, \quad j = 1, 2, 3, \dots$$

From (4.46) and Rellich's theorem, we can find subsequence $\{f_{j_s}\}_{s=1}^\infty$, $1 \leq j_1 < j_2 < \dots$, $f_{j_s} \rightarrow f$ in $L^2(X)$. Since $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$, we get a contradiction. Thus, $\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P$ is finite dimensional. Let $\{f_1, f_2, \dots, f_d\}$ be an orthonormal frame of $\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P$, $d < \infty$. As (4.46), we have $f_j = F f_j$, $j = 1, 2, \dots, d$. Thus, $f_j \in C^\infty(X)$, $j = 1, 2, \dots, d$, and hence $\hat{\mathcal{P}}^\perp \bigcap \text{Ker } P \subset C^\infty(X)$.

We can repeat the procedure above and conclude that $\hat{\mathcal{P}}_0^\perp \bigcap \text{Ker } P \subset C^\infty(X)$, $\hat{\mathcal{P}}_0^\perp \bigcap \hat{\mathcal{P}} \subset C^\infty(X)$, $\hat{\mathcal{P}}_0^\perp \bigcap \text{Ker } P, \hat{\mathcal{P}}_0^\perp \bigcap \hat{\mathcal{P}}$ are all finite dimensional. \square

5. SPECTRAL THEORY FOR P

In this section, we will prove Theorem 1.7. For any $\lambda > 0$, put

$$\Pi_{[-\lambda, \lambda]} := E([-\lambda, \lambda]),$$

where E denotes the spectral measure for P (see section 2 in Davies [5], for the precise meaning of spectral measure). We need

Theorem 5.1. *Fix $\lambda > 0$. We have $P \Pi_{[-\lambda, \lambda]} \equiv 0$ on X .*

Proof. As before, let N be the partial inverse of P and let Π be the orthogonal projection onto $\text{Ker } P$. We have

$$(5.1) \quad NP + \Pi = I.$$

From (5.1), we have

$$(5.2) \quad NP^2 \Pi_{[-\lambda, \lambda]} = P \Pi_{[-\lambda, \lambda]}.$$

From (1.5), (5.2) and notice that

$$P^2 \Pi_{[-\lambda, \lambda]} : L^2(X) \rightarrow L^2(X) \text{ is continuous,}$$

we conclude that

$$(5.3) \quad P \Pi_{[-\lambda, \lambda]} : L^2(X) \rightarrow H^2(X) \text{ is continuous.}$$

Similarly, we can repeat the procedure above and deduce that

$$(5.4) \quad P^2 \Pi_{[-\lambda, \lambda]} : L^2(X) \rightarrow H^2(X) \text{ is continuous.}$$

From (5.4), (5.2) and (1.5), we get

$$P \Pi_{[-\lambda, \lambda]} : L^2(X) \rightarrow H^4(X) \text{ is continuous.}$$

Continuing in this way, we conclude that

$$(5.5) \quad P \Pi_{[-\lambda, \lambda]} : L^2(X) \rightarrow H^m(X) \text{ is continuous, } \forall m \in \mathbb{N}_0.$$

Note that $P \Pi_{[-\lambda, \lambda]} = \Pi_{[-\lambda, \lambda]} P = (P \Pi_{[-\lambda, \lambda]})^*$, where $(P \Pi_{[-\lambda, \lambda]})^*$ is the adjoint of $P \Pi_{[-\lambda, \lambda]}$. By taking adjoint in (5.5), we get

$$\Pi_{[-\lambda, \lambda]} P = P \Pi_{[-\lambda, \lambda]} : H^{-m}(X) \rightarrow L^2(X) \text{ is continuous, } \forall m \in \mathbb{N}_0.$$

Hence,

$$(5.6) \quad (P \Pi_{[-\lambda, \lambda]})^2 = P^2 \Pi_{[-\lambda, \lambda]} : H^{-m}(X) \rightarrow H^m(X) \text{ is continuous, } \forall m \in \mathbb{N}_0.$$

From (5.6), (5.2) and (1.5), the theorem follows. \square

We need

Theorem 5.2. *For any $\lambda > 0$, $\Pi_{[-\lambda, \lambda]} \equiv \Pi$ on X .*

Proof. From (5.1) and Theorem 5.1, we get

$$(5.7) \quad \Pi \Pi_{[-\lambda, \lambda]} \equiv \Pi_{[-\lambda, \lambda]} \text{ on } X.$$

On the other hand, it is clearly that $\Pi \Pi_{[-\lambda, \lambda]} = \Pi$. From this observation and (5.7), the theorem follows. \square

Now, we can prove

Theorem 5.3. *Spec P is a discrete subset in \mathbb{R} and for every $\lambda \in \text{Spec P}$, $\lambda \neq 0$, λ is an eigenvalue of P and the eigenspace*

$$H_\lambda(\text{P}) := \{u \in \text{Dom P} ; \text{P} u = \lambda u\}$$

is a finite dimensional subspace of $C^\infty(X)$.

Proof. Since P has L^2 closed range, there is a $\mu > 0$ such that $\text{Spec P} \subset]-\infty, -\mu] \cup [\mu, \infty[$. Fix $\lambda > \mu$. Put $\Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} := E([-\lambda, -\mu] \cup [\mu, \lambda])$. Note that

$$\Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} = \Pi_{[-\lambda, \lambda]} - \Pi_{[-\frac{\mu}{2}, \frac{\mu}{2}]}.$$

From this observation and Theorem 5.2, we see that

$$(5.8) \quad \Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} \equiv 0.$$

We claim that $\text{Spec P} \cap \{[-\lambda, -\mu] \cup [\mu, \lambda]\}$ is discrete. If not, we can find $f_j \in \text{Rang } E([-\lambda, -\mu] \cup [\mu, \lambda])$, $j = 1, 2, \dots$, with $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$. Note that

$$f_j = \Pi_{[-\lambda, -\mu] \cup [\mu, \lambda]} f_j, \quad j = 1, 2, \dots$$

From this observation, (5.8) and Rellich's theorem, we can find subsequence $\{f_{j_s}\}_{s=1}^\infty$, $1 \leq j_1 < j_2 < \dots$, $f_{j_s} \rightarrow f$ in $L^2(X)$. Since $(f_j | f_k) = \delta_{j,k}$, $j, k = 1, 2, \dots$, we get a contradiction. Thus, $\text{Spec P} \cap \{[-\lambda, -\mu] \cup [\mu, \lambda]\}$ is discrete. Hence Spec P is a discrete subset in \mathbb{R} .

Let $r \in \text{Spec P}$, $r \neq 0$. Since Spec P is discrete, $\text{P} - r$ has L^2 closed range. If $\text{P} - r$ is injective, then $\text{Range}(\text{P} - r) = L^2(X)$ and

$$(\text{P} - r)^{-1} : L^2(X) \rightarrow L^2(X)$$

is continuous. We get a contradiction. Hence r is an eigenvalue of Spec P. Put

$$H_r(\text{P}) := \{u \in \text{Dom P} ; \text{P} u = ru\}.$$

We can repeat the procedure above and conclude that $\dim H_r(\text{P}) < \infty$. Take $0 < \mu_0 < \lambda_0$ so that $r \in \{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]\}$. From Theorem 5.2, we see that

$$\Pi_{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]} \equiv 0.$$

Since

$$H_r(\text{P}) = \{\Pi_{[-\lambda_0, -\mu_0] \cup [\mu_0, \lambda_0]} f; f \in H_r(\text{P})\},$$

$H_r(\text{P}) \subset C^\infty(X)$. The theorem follows. \square

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